# How Graph Theory Links Topological Quantum Field Theory to Homotopy Transfer

Leon Menger

## Contents

## Introduction

In this exposé I want to give an introduction to the usage of graph theory in (Topological) Quantum Field Theory using the example of $BF$ theory. In particular I want to highlight how it can provide a link between the notion of effective actions and the ideas of homotopy transfer of algebraic structures. Effective actions arise in mathematical physics as "practical, smaller, but in some sense equivalent" models of gauge theories.
Using the example of topological $BF$ theory I will explain a well-understood case of this analogy, aiming to sketch generalisable ideas where possible. To complement this, I will talk about the result of Kontsevich–Soibelman [KS01] which links homotopy transfer to the same type of sum-over-trees formulae that can arise from effective actions as Feynman diagrams.

The style of this exposé – in accordance with its length – will be rather casual and focus on the bigger-picture ideas rather than the extensive technicalities at work in the background. By injecting remarks and associated references whenever natural, it will be more a survey than a technical paper. The paper can be read[1] by people with various levels of familiarity with the subject, as anything between a first foray into the matter and a brief sketch of a familiar story.

The reader familiar with the idea of gauge-fixing in the BV formalism and $BF$ theory can completely skip Section 1 or skim it to refresh some notions. For readers with no familiarity with the topics, Section 1 should be read and taken as a first "appetiser" only. To read Section 2, grasping the overarching concepts is most important. For details the reader is referred to the cited sources.

[1] Hopefully

# 1 Gauge Fixing

## 1.1 The Gauge Problem

First we want to take a step back to briefly introduce the concept of gauge-fixing, particularly its incarnation in the BV formalism. Let in the following $S_{cl}\colon \mathcal{F}_{cl} \longrightarrow \mathbb{R}$ denote the *classical* action functional on the space of classical fields $\mathcal{F}_{cl}$. In its Lagrangian incarnation it takes the form

$$S_{cl}(\varphi_i) = \int_M \mathcal{L}_{cl}(\varphi_i), \tag{1}$$

where $\mathcal{L}_{cl}$ is a map from $\mathcal{F}_{cl}$ to top-forms on $M$. Actions can be used to describe various quantities from topological invariants on manifolds to surface areas of certain geometric objects and even complex dynamical systems and physical theories like gravity, electromagnetism or the standard model of particle physics. Solutions to such problems are captured by certain elements in $\mathcal{F}_{cl}$ that satisfy the **classical master equation** (CME)

$$\{S_{cl}, S_{cl}\} = 0. \tag{CME}$$

However in most cases the usage of action functionals suffers from a simple problem: The data used to describe a classical field theory usually far exceeds the minimal data one would *need* to describe the quantities it is supposed to capture. This manifests as invariances of the action and solutions to the (CME) under certain transformations of the fields. In sufficiently nice theories these transformations can be described as the action of a (symmetry) group. These groups are hence called the "gauge group" of a theory.

Such redundancies pose a particularly big problem if one wants to make sense of expressions of the following type:

$$\int_{\mathcal{F}_{cl}} e^{\frac{i}{\hbar} S_{cl}} \ . \tag{2}$$

Here $\hbar$ is some formal parameter, $S_{cl}$ denotes the classical action and $\mathcal{F}_{cl}$ the classical space of fields. Such expressions are initially motivated from physics where – in *quantum* field theory – one needs to calculate "weighted integrals over all possible field configurations" to obtain the resulting dynamics on smaller scales. This should be seen as a kind of "generalised expectation value of how dynamics play out". In physically motivated scenarios the formal parameter $\hbar$ is the reduced Planck's constant. As an additional problem, the space $\mathcal{F}_{cl}$ is often infinite-dimensional and expressions like Equation (2) are simply ill-defined.

Formally one can try to make sense of such integrals using the stationary phase formula (see e.g. [Mne17, Theorem 3.48, Theorem 4.5]). While for finite-dimensional $\mathcal{F}_{cl}$ equivalence is granted by a theorem, in the infinite-dimensional case one takes it as a heuristically motivated definition to make sense of the a priori ill-defined expression (2). The redundancies imply however that the Hessian of $S_{cl}$ is degenerate at stationary points, making even this perturbative expansion ill-defined.

This is where the **problem** in "gauge problem" really arises. A particularly nice way to approach it, is the BV(-BRST) formalism. It provides a constructive method of extending a classical theory with extra fields, such that all redundancies are in a sense part of the action. For comprehensive introductions to the basics of the BV formalism and enhancements see [CMR14; CM20]. In particular [Mne17] gives a good chronological introduction to the subject.

To give a rough idea, the BV formalism allows us to work on the level of individual representants of field configurations instead of just coarse equivalence classes. This can be summed up into an extension of the (CME) to the **quantum master equation**

$$\Delta e^{\frac{i}{\hbar}S} = 0 \qquad \Longleftrightarrow \qquad \frac{1}{2}\{S, S\} - i\hbar\Delta S = 0, \tag{QME}$$

which a BV theory has to obey.

## 1.2 The BV Integral

Assume now we have a suitable BV action $S$ satisfying the (QME). Since it contains the redunancies as additional fields, the idea is to choose particularly "nice" representants for them, to make the stationary phase formula for (2) well-defined. Using the redundancies in this way is a concept found across various approaches to dealing with the gauge problem in field theories.

Denote the space of BV fields by $\mathcal{F}_{BV}$. It is obtained as $T^*[-1]\mathcal{F}_{BRST}$, where $\mathcal{F}_{BRST}$ should be seen as $\mathcal{F}_{cl}$ + *"symmetries"*. E.g. for $BF$ theory we can write

$$\begin{aligned} \mathcal{F}_{BV} &= \Omega^\bullet(M; \mathfrak{g})[1] \oplus \Omega^\bullet(M; \mathfrak{g}^*)[\dim(M) - 2] && (3) \\ &= T^*[-1](\Omega^\bullet(M; \mathfrak{g})[1]). && (4) \end{aligned}$$

As a result $\mathcal{F}_{BV}$ has the structure of an odd-symplectic space with $(-1)$-symplectic form $\Omega_{BV}$ given by the canonical coordinate symplectic form.

Assume now that our space of BV fields arises from some differential graded Lie algebra $\mathfrak{h}$ as $\mathcal{F}_{BV} := T^*[-1](\mathfrak{h}[1])$[2]. Whenever there is a split of $\mathfrak{h}$ into subcomplexes $\mathfrak{h}' \oplus \mathfrak{h}''$ where $\mathfrak{h}''$ is acyclic, i.e. contributes nothing to the cohomology of $\mathfrak{h}$, we get a corresponding split

[2]As is the case for $BF$ theory, see above.

$\mathcal{F}_{BV} = \mathcal{F}' \oplus \mathcal{F}''$. Our goal is to dismiss the degrees of freedom in $\mathcal{F}''$ since they have no impact on the cohomology underlying our solutions to the (QME). Choosing a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}''$ lets us do exactly that by means of the **BV integral**

$$e^{\frac{i}{\hbar}S_{\text{eff}}} = \frac{1}{N}\int_{\mathcal{L}} e^{\frac{i}{\hbar}S_{BV}}\mu_{\mathcal{L}}, \tag{5}$$

where $N$ is some normalisation factor and $\mu_{\mathcal{L}}$ denotes a volume form on $\mathcal{L}$. $S_{\text{eff}}$ is called the **effective action** resulting from the gauge-fixing procedure and it is a function on $\mathcal{F}'$[3] Terms in $S_{BV}$ that live only in $\mathcal{F}'$ are seen as constants, terms only in $\mathcal{F}''$ as purely on $\mathcal{L}$ and mixed terms as perturbations yielding additional terms of the effective actions. Thus the BV integral in Equation (5) gives us a *perturbative* definition of $S_{\text{eff}}$. Nicely enough, this procedure guarantees that $S_{\text{eff}}$ satisfies the (QME).

Of course it is not clear a priori how to construct Lagrangians in $\mathcal{F}_{BV}$. Much less “useful” Lagrangians that help aleviate the degeneracies in the stationary phase formular. We will formulate a special class of Lagrangians in Section 2.2 when talking about effective actions stemming from $BF$ theory.

### 1.3 $BF$ Theory in the BV Formalism

$BF$ theory is a special type of topological field theory since it can be formulated on manifolds $M$ of any positive dimension, which do not need to be orientable and can have a boundary. Its classical action functional is

$$S_{BF} = \int_M \left\langle B, dA + \frac{1}{2}[A,A]_{\mathfrak{g}} \right\rangle, \tag{6}$$

where $A \in \Omega^1(M;\mathfrak{g})$ is a connection 1-form of some principal $G$-bundle $P$ such that $\mathfrak{g} = Lie(G)$ is some semisimple Lie algebra, $B \in \Omega^{n-2}(M;\mathfrak{g}^*)$ and $\langle -,- \rangle$ denotes the Killing form pairing. The classical equations of motion – i.e. solutions to (CME) – amount to

$$0 = dA + \frac{1}{2}[A,A]_{\mathfrak{g}} = F_A, \tag{7}$$

$$0 = d_A B. \tag{8}$$

I.e. the space of classical solution is described by flat connections $A$ and fields $B$ with vanishing covariant derivative. In low dimensions BF theory has intricate links to topological invariants of the underlying manifold [Wit89; CCFM95; PWY17]. It also has considerable physical relevance since it can be deformed to obtain theories of gravity [Mik06; FS12]. Gravitational theories obtained in this way are commonly used in quantisation schemes like

[3] One could see $S_{\text{eff}}$ as a result of taking a normalised expectation value over all fields in $\mathcal{F}''$.

Loop Quantum Gravity [RS06; DL11].

A particularly interesting feature of $BF$ theory is that its full BV theory – taking into account all symmetries and their redundancies – can be brought into the same form as its classical version using superfields [CR01]:

$$S_{BF}^{BV} = \int_M \left\langle B, dA + \frac{1}{2}[A, A] \right\rangle, \qquad \mathcal{F}_{BV} = T^*[-1](\Omega^\bullet(M; \mathfrak{g})[1]). \tag{9}$$

Hence from now on we will always use $S_{BF}$ to denote the full BV $BF$ action.

For the purpose of this article it is helpful to see $S_{BF}$ as a generating function for structure constants: Consider the triple $\mathfrak{h} := (\Omega^\bullet(M; \mathfrak{g}), d, [-,-])$, where $d$ is the deRham differential and $[-,-]$ is the Lie bracket induced by $[-,-]_{\mathfrak{g}}$. The (CME) is equivalent to

$$\langle B, d^2 A\rangle + \left\langle B, [A, dA] + \frac{1}{2}d[A, A] \right\rangle + \left\langle B, \frac{1}{2}[[A, A], A] \right\rangle = 0. \tag{10}$$

Since the terms of different powers in $A$ have to vanish separately we get the **differential condition** $d^2 = 0$ from the linear, the **Leibniz identity** from the quadratic and the **Jacobi identity** from the cubic terms. This tells us that the pair of operations $(d, [-,-])$ promotes $\Omega^\bullet(M; \mathfrak{g})$ to a differential graded Lie algebra. In other words we should see $BF$ theory in the BV formalism as the generating function for dg Lie algebra structures. More appropriately however we should inspect the (QME) since we are working with a full BV theory. Apart from the classical terms above it also yields the independent relation

$$-i\hbar \left(\mathrm{Str}_{\mathfrak{h}}(d(-)) + \mathrm{Str}_{\mathfrak{h}}([A, -])\right) = 0. \tag{11}$$

Since $d$ is of degree 1, $\mathrm{Str}_{\mathfrak{h}}(d(-)) = 0$. Thus the above condition boils down to $\mathrm{Str}_{\mathfrak{h}}([A, -]) = 0$, which is the **unimodularity condition** for Lie algebras. Thus in the following we will restrict to unimodular Lie algebras to guarantee that (QME) can be satisfied.

## 2 Two Views on Homotopy Transfer

### 2.1 Classical Homotopy Transfer

In the following we will briefly state a result about the homotopy transfer of differential graded Lie algebras. It goes back to earlier works of [GS86] and [Mer01], however we use here the convenient version found in [KS01, p. 6.4] since it explicitly links the result to a "sum over trees".

First we need to set the stage. Let's assume we have two cochain complexes. A natural question to ask is then what data is needed to say that one complex is a subcomplex of the other, such that it still "captures the same cohomology". In other words, we are asking for the correct definition of a quasi-isomorphism:

**Definition 2.1** (IPK-Triples) Let $(V, d)$ and $(W, d')$ be cochain complexes and let $(\imath, p, K)$ be a triple such that:

- **I)** $\imath\colon W \longrightarrow V$ and $p\colon V \longrightarrow W$ are chain maps such that $p \circ \imath = \mathrm{id}$.
- **II)** $K\colon V^\bullet \longrightarrow V^{\bullet-1}$ is a chain homotopy between $p \circ \imath$ and the identity, i.e.

$$\mathrm{id} - \imath \circ p = d \circ K + K \circ d. \tag{12}$$

- **III)** $K^2 = 0$, $K \circ \imath = 0$, $p \circ K = 0$.

We call such data a **contraction of** $V$ **onto** $W$ and the triple $(\imath, p, K)$ an **IPK-triple**.

IPK-triples can be depicted diagramatically as

$$K \circlearrowright (V, d, [-,-]) \underset{\imath}{\overset{p}{\rightleftarrows}} (W, d', [-,-])$$

The data of an IPK-triple is a slightly enhanced version of the usual notion of quasi-isomorphisms between cochain complexes: The three additional conditions in **III)** essentially state that $K$ maps into **and** annihilates the complex we contract onto:

$$
\begin{array}{ccccccc}
\vdots & & & \dots & & \\
k+1 & (V\setminus \imath(W))^{k+1} & & \imath(W)^{k+1} & & \\
 & & \searrow^{K} & & \searrow^{K} & \\
k & (V\setminus \imath(W))^{k} & & \imath(W)^{k} & & 0 \\
 & & \searrow^{K} & & \searrow^{K} & \\
k-1 & (V\setminus \imath(W))^{k-1} & & \imath(W)^{k-1} & & 0 \\
\vdots & & & \dots & &
\end{array}
$$

Assume now we are given an IPK-triple as above. Now however, the base dg vector space $(V, d)$ is also equipped with a compatible Lie bracket $[-,-]$, making it a differential graded Lie algebra. One could then wonder if the data of an IPK-triple is enough to enhance quasi-isomorphisms on the level of cochain complexes to a bigger algebraic context. Indeed there is a very constructive result:

**Theorem 2.2** (Homotopy Transfer by Sum-over-Trees [KS01][4]) Let $(\mathfrak{h}, d, [-,-])$ be a differential graded Lie algebra and let $(\imath, p, K)$ be an IPK-triple contracting $\mathfrak{h}$ onto $\mathfrak{f}$. Then $\mathfrak{f}$ can be equipped with the structure of an $L_\infty$ algebra[5] $(\mathfrak{f}, \{l_i\})$ by defining the totally antisymmetric maps

$$l_1 := d^{\mathfrak{f}} = p \circ d \circ \imath, \tag{13}$$

$$l_2 := p \circ [-,-] \circ (\imath \otimes \imath), \tag{14}$$

$$l_n := \sum_{T \in \mathrm{obTree}(n)} l(T), \qquad n \geq 3. \tag{15}$$

The sum for the $l_i$ maps goes over oriented, planar, binary rooted trees $T$ with $n$ leaves. Their constituent maps $l(T)$ are formed as follows: Decorate all leaves of $T$ by $\imath$ and all internal vertices by $[-,-]$. Then decorate all internal edges by $K$ and the root of the tree by $p$. Concatenation of these operations from the leaves to the root yields $l(T)$.

The following illustration shows two examples for operations $l(T)$ contributing to $l_4$:

[4] Actually [KS01, p. 6.4] is concerned with $A_\infty$ algebras and states a more general version. We state here the $L_\infty$ version, restricted to dgLA's.

[5] The definition of an $L_\infty$ algebra can be found e.g. in [Mne08], however the relevant structure equations are also found down below, named ($L_\infty$-relations).

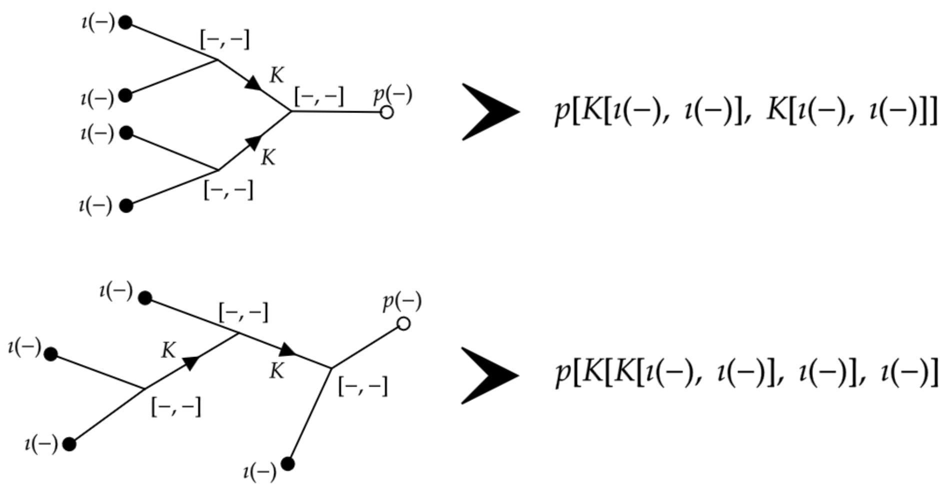



**Figure 1:** Illustration of two graphs contributing to $l_4$. Leaves are denoted by bullets, the root vertex by a hollow bullet. Internal edges are decorated with an arrow to indicate the orientation.

Really we should see Theorem 2.2 as a result about $L_\infty$ algebras (or $A_\infty$, see [KS01]). Since every dgLA is in particular an $L_\infty$ algebra with vanishing operations for $n \geq 3$, Theorem 2.2 tells us that the quasi-isomorphism on the level of cochain complexes lifts to a quasi-isomorphism of specific $L_\infty$ algebras over the same bases.

The occurence of **binary** rooted trees is due to the fact that we only have operations with either 1 or 2 inputs in a dgLa. One should see the "sum over trees" defining the operations $l_n$ in the subcomplex as a way of including every possible way to combine a given number of inputs using the operations. We will rediscover such combinations when looking at effective $BF$ theory in the following section.

## 2.2 Effective $BF$ Theory

In Section 1.2 we left off in a rather awkward situation. Having talked about the gauge problem and the BV integral, we had to concede that – at this point – there were no clear-cut Lagrangians that provide nice examples of gauge-fixing and thus effective actions.

So let us go back to spaces of BV fields of the form $\mathcal{F}_{BV} := T^*[-1](\mathfrak{h}[1])$, for some (unimodular) dg Lie algebra $\mathfrak{h}$. Let us further assume that $\mathfrak{h}$ is **nilpotent**, i.e. $\mathfrak{h}$ has vanishing Killing form[6]. Luckily a particularly nice class of Lagrangians can be constructed from the

[6]This leads to drastic simplifications of the terms we need to consider for $S_{\text{eff}}$. In particular we then recover exactly the homotopy transfer result of Theorem 2.2 and not some stronger result on unimodular $L_\infty$ algebras. A good source showing how these more general structures are recovered from $BF$ theory for more general dg Lie algebras is [Mne08, sec. 4].

data of an IPK-triple $(\imath, p, K)$ contracting $\mathfrak{h}$ onto $\mathfrak{f}$. Using the triple we split the space of BV fields as

$$\mathcal{F}_{BV} := T^*[-1](\mathfrak{h}[1]) = \underbrace{\mathcal{F}'}_{\text{From } \mathfrak{f}} \oplus \underbrace{\mathcal{F}''}_{\text{From complement}}. \tag{16}$$

$K$ then uniquely defines a Lagrangian submanifold of $\mathcal{F}''$ via

$$\mathcal{L}_K := \ker(K)[1] \oplus \operatorname{im}(K^*)[-2]. \tag{17}$$

For a given action $S_{BV}$ formulated on $\mathcal{F}_{BV}$ we can now make sense of the effective action. Its perturbative definition as a formal power series in $\hbar$ is given by the BV integral (5)

$$e^{\frac{i}{\hbar}S_{\text{eff}}} = \frac{1}{N} \int_{\mathcal{L}_K} e^{\frac{i}{\hbar}S_{BV}} \mu_{\mathcal{L}_K}. \tag{18}$$

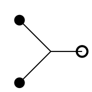
While one could use the stationary phase formula to calculate the explicit form of $S_{\text{eff}}$ by hand, we want to go a more intuitive route here.

In the following, we work with the BV $BF$ action $S_{BF}$ which can be seen as a generating function for the structure equations of a differential graded Lie algebra (see Section 1.3). From the splitting of the space of fields into $\mathcal{F}_{BV} = \mathcal{F}' \oplus \mathcal{F}''$ we obtain a splitting of fields

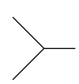
$$A = A' + A'', \qquad B = B' + B''. \tag{19}$$

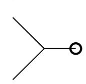
The effective action is obtained by integration over $\mathcal{L}_K \subset \mathcal{F}''$. For the purpose of integration fields $A', B'$ are constants and fields $A'', B''$ are integrated over. In the following we decorate constant fields by *external vertices* and non-constant fields by *half-edges*. This yields the following diagramatic building blocks, called **Feynman rules**:

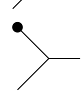
$$S_{BF} = \int_M \langle B', dA' \rangle \implies \tag{20}$$

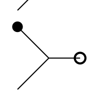

$$+\left\langle B', \frac{1}{2}[A', A'] \right\rangle \implies \tag{21}$$


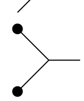
$$+\langle B'', dA'' \rangle \implies \tag{22}$$


$$+\left\langle B'', \frac{1}{2}[A'', A''] \right\rangle \implies \tag{23}$$


$$+\left\langle B', \frac{1}{2}[A'', A''] \right\rangle \implies \tag{24}$$


$$+\langle B'', [A', A''] \rangle \implies \tag{25}$$


$$+\langle B', [A', A''] \rangle \implies \tag{26}$$


$$+\left\langle B'', \frac{1}{2}[A', A'] \right\rangle \implies \tag{27}$$


The graphs above are oriented and read left-to-right. Incoming vertices/half-edges are decorated by $\imath(A'), A''$. The fields $\imath(B'), B''$ decorate outgoing vertices/half-edges respectively. The only exceptions are the building blocks $B'\bullet\!\!-\!\!\blacktriangleright\!\!-\!\!\circ A'$ and $B''\!-\!\!\blacktriangleright\!\!-A''$ which send a $B'$ vertex to a $A'$ vertex and a $B''$ half-edge to a $A''$ half-edge respectively. The term $B''\!-\!\!\blacktriangleright\!\!-A''$ is enabling us to construct more complex graphs from the above building blocks in the first place.

The effective action is now obtained from the Feynman rules as follows[7]:

i) Decorate the **all-edge** building block $B''\!-\!\!\blacktriangleright\!\!-A''$ with the chain homotopy $K$.

ii) Decorate **incoming** vertices by $\imath(A')$ and half-edges by $A''$.

iii) Decorate **outgoing** vertices by $\langle B', p(-)\rangle$ and half-edges by $B''$.

iv) Decorate **internal** vertices with the Lie bracket $[-,-]$.

v) Build any oriented graph by fitting the building blocks at equally decorated half-edges.

Naturally the resulting graphs should contribute to $S_{\text{eff}}$ only up to automorphism. Denoting the above decoration rules by $\Phi$ this leads to the following effective action:

$$S_{\text{eff}} = \sum_{n\geq 1}\sum_{\Gamma\in\text{obTree}(n)} \frac{1}{|\operatorname{Aut}(\Gamma)|}\Phi(\Gamma) \tag{28}$$

$$= \langle B', d'A'\rangle + \left\langle B', \frac{1}{2}p\left([\imath(A'),\imath(A')]\right)\right\rangle + \frac{1}{2}\left\langle B', p\left([K[\imath(A'),\imath(A')],\imath(A')]\right)\right\rangle \tag{29}$$

$$+ \frac{1}{2}\left\langle B', p\left([K[K[\imath(A'),\imath(A')],\imath(A')],\imath(A')]\right)\right\rangle \tag{30}$$

$$+ \frac{1}{8}\left\langle B', p\left([K[\imath(A'),\imath(A')],K[\imath(A'),\imath(A')]]\right)\right\rangle + \dots \tag{31}$$

Maybe to no surprise, there will be no pictorial representations of these graphs, because we already have them: The reader will certainly agree that the form of the terms in the effective action and the graph-induced operations in Figure 1 is *suspiciously* similar. We will see just how deep this similarity goes in the next and last section.

### 2.3 The Link

As noted above, the terms in the effective action look similar to the ones we found in the sum-over-trees formula from Theorem 2.2 depicted in Figure 1. Looking back this *might*

[7] Note that we slightly abuse notation here to denote the fields of $S_{\text{eff}}$ by primed fields just as we did for $\mathcal{F}' \subset \mathcal{F}_{BV}$. This serves to further strengthen the idea of a quasi-isomorphic subspace and helps to support the parallel to the Kontsevich–Soibelman result on $L_\infty$ algebras.

not even come as a surprise. After all we can read the $BF$ action as a generating function for (unimodular) dg Lie algebras, which are exactly the subject of the homotopy transfer described in Section 2.1. We even used the exact same data – an IPK-triple – for homotopy transfer and to obtain an effective action of $BF$ theory. Spinning this further, we might expect to find the same relations $l_i$ from Theorem 2.2 hidden in the effective action. Indeed we can immediately write

$$S_{\text{eff}}(A', B') = \sum_{n\geq 1} \sum_{\Gamma \in \text{obTree}(n)} \frac{1}{|\operatorname{Aut}(\Gamma)|} \Phi(\Gamma) = \sum_{n \geq 1} \int_M \frac{1}{n!} \langle B', l_n(\underbrace{A', \ldots, A'}_{n\text{-times}}) \rangle. \tag{32}$$

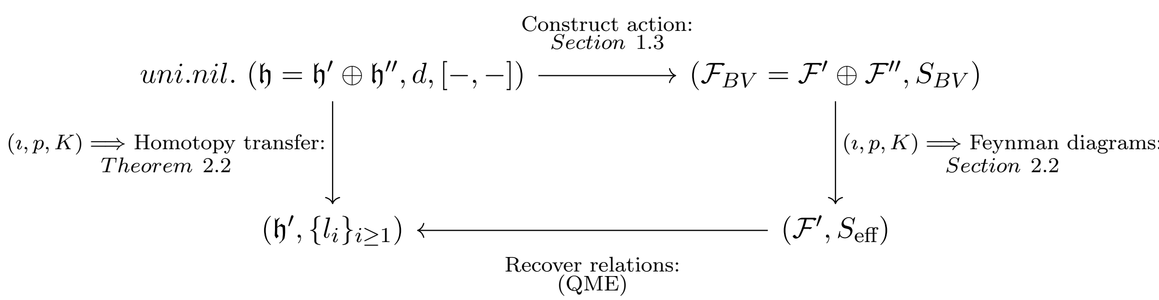
All that is left is checking if $S_{\text{eff}}$ is indeed a generator for the structure of an $L_\infty$ algebra. Usual $BF$ theory recovers the relations of a (unimodular) dg Lie algebra by means of the (QME). Since the BV integral ensures that $S_{\text{eff}}$ also satisfies the QME, we might expect the same to happen here. Indeed the (QME) yields for all $n \geq 1$:

$$\sum_{r+s=n} \frac{1}{r!s!} l_{r+1}(A', \ldots, A', l_s(A', \ldots, A')) = 0, \qquad (L_\infty\text{-relations})$$

$$\frac{1}{n!} \operatorname{Str}_{\mathfrak{h}'}(l_{n+1}(A', \ldots, A', -)) = 0. \qquad \text{(Higher unimodularity relations)}$$

Thus we should see $S_{\text{eff}}$ as a generating function for the structure of an $L_\infty$ algebra. The additional higher unimodularity relations are due to the fact that we started off with a unimodular dg Lie algebra to ensure that (QME) is satisfied.

Summing up the results of this exposé we can draw the following (commutative) diagram between the sum-over-trees result by Kontsevich–Soibelman Theorem 2.2 and the effective action of $BF$ theory obtained from an IPK-triple described in Section 2.2:



## References


[CCFM95] A. S. Cattaneo, P. Cotta-Ramusino, J. Fröhlich, and M. Martellini. "Topological BF theories in 3 and 4 dimensions". In: *Journal of Mathematical Physics* 36.11 (Nov. 1995), pp. 6137–6160.

[CMR14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. "Classical BV Theories on Manifolds with Boundary". In: *Communications in Mathematical Physics* 332.2 (2014), pp. 535–603.

[CM20] A. S. Cattaneo and N. Moshayedi. "Introduction to the BV-BFV formalism". In: *Reviews in Mathematical Physics* 32.09 (2020), p. 2030006.

[CR01] A. S. Cattaneo and C. A. Rossi. "Higher-Dimensional BF Theories in the Batalin-Vilkovisky Formalism: The BV Action and Generalized Wilson Loops". In: *Communications in Mathematical Physics* 221.3 (Aug. 2001), pp. 591–657.

[DL11] M. Dupuis and E. R. Livine. "Revisiting the simplicity constraints and coherent intertwiners". In: *Classical and Quantum Gravity* 28.8 (2011), p. 085001.

[FS12] L. Freidel and S. Speziale. "On the Relations between Gravity and BF Theories". In: *Symmetry, Integrability and Geometry: Methods and Applications* 32 (8 2012).

[GS86] V. Gugenheim and J. Stasheff. "On perturbations and $A_\infty$-structures". In: *Bulletin de la Société Mathématique de Belgique. Série A* 38 (Jan. 1986).

[KS01] M. Kontsevich and Y. Soibelman. *Homological mirror symmetry and torus fibrations*. 2001. arXiv: `math/0011041 [math.SG]`.

[Mer01] S. A. Merkulov. *Strongly homotopy algebras of a Kähler manifold*. 2001. arXiv: `math/9809172 [math.AG]`.

[Mik06] A. Mikovic. "Quantum Gravity as a Broken Symmetry Phase of a BF Theory". In: *Symmetry, Integrability and Geometry: Methods and Applications* 86 (2 2006).

[Mne08] P. Mnev. *Discrete BF theory*. 2008. arXiv: `0809.1160 [hep-th]`.

[Mne17] P. Mnev. *Lectures on Batalin-Vilkovisky formalism and its applications in topological quantum field theory*. 2017.

[PWY17] P. Putrov, J. Wang, and S.-T. Yau. "Braiding statistics and link invariants of bosonic/fermionic topological quantum matter in 2+1 and 3+1 dimensions". In: *Annals of Physics* 384 (Sept. 2017), pp. 254–287.

[RS06] C. Rovelli and S. Speziale. "On the expansion of a quantum field theory around a topological sector". In: *General Relativity and Gravitation* 39.2 (2006), pp. 167–178.

[Wit89] E. Witten. “Quantum field theory and the Jones polynomial”. In: *Communications in Mathematical Physics* 121.3 (1989), pp. 351–399.